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Cryptogauge symmetry and cryptoghosts for crypto-Hermitian Hamiltonians

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Received 25 July 2007, in final form 14 November 2007

Published 3 June 2008

Online at stacks.iop.org/JPhysA/41/244026

Abstract

We discuss the Hamiltonian $H = p^2/2 - (ix)^{2n+1}$ and the mixed Hamiltonian $H_{\text{mixed}} = (p^2 + x^2)/2 - g(ix)^{2n+1}$. The Hamiltonians H and in some cases also H_{mixed} are crypto-Hermitian in a sense that, in spite of their apparent non-Hermiticity, a quantum spectral problem can be formulated such that the spectrum is real. We note that the corresponding classical Hamiltonian system can be treated as a gauge system, with the imaginary part of the Hamiltonian playing the role of the first class constraint. Several different nontrivial quantum problems can be formulated on the basis of this classical problem. We formulate and solve some such problems. We consider then the mixed Hamiltonian and find that its spectrum undergoes in certain cases a rather amazing transformation when the coupling g is sent to zero. There is an infinite set of exceptional points $g_{\star}^{(j)}$ where a couple of eigenstates of H coalesce and their eigenvalues cease to be real. When quantization is done in the most natural way such that gauge constraints are imposed on quantum states, the spectrum should not be positive definite, but must involve the negative energy states (ghosts). We speculate that, in spite of the appearance of ghost states, unitarity might still be preserved.

PACS numbers: 03.65.Ca, 03.65.Ge

1. Introduction

For certain apparently complex Hamiltonians, the spectral problem can be formulated such that the spectrum has a perfectly ‘normal’ form with bounded from below real energies. Such Hamiltonians can thus be called ‘crypto-Hermitian’ or ‘cryptoreal’. Apparently, such crypto-Hermitian Hamiltonians were first discussed in association with Reggeon field theory back in 1976 [1]. Somewhat later, crypto-Hermitian Hamiltonians were considered by mathematicians in a more habitual Schrödinger setup. Gasymov observed that the Schrödinger operator with

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certain complex periodic potentials, like $V(x) = e^{ix}$, has a real spectrum [2]. In [3], it was proved that the spectrum of the Hamiltonian with complex potential $V(x) = x^2 + i\beta x^3$ is real for small enough β . General properties of crypto-Hermitian (or *quasi-Hermitian* as the authors called this property) operators were studied in [4].

Before going further, a comment on the terminology is in order. At the moment, there is no unique generally adopted name for the Hamiltonians of this kind. Besides *quasi-Hermitian*, the term *pseudo-Hermitian* is also often used. A Hamiltonian is usually called pseudo-Hermitian if it satisfies the property

$$H^\dagger = \eta H \eta^{-1} \tag{1}$$

with some Hermitian invertible η . However, this does not guarantee yet that the spectrum is real. To this end, the operator η should be representable as [5]

$$\eta = O^\dagger O \tag{2}$$

or, equivalently, the norm $\langle \psi | \eta \psi \rangle$ should be positive definite for any nonzero Hilbert space vector ψ [4, 6]. Anyway, the semantics of the words *quasi-Hermitian* or *pseudo-Hermitian* is ‘not quite Hermitian’ with a flavor of inferiority, ‘second-rankness’ compared to *Hermitian*. For example, pions are pseudo-Goldstone particles meaning that they are *not* Goldstone particles. But we want to emphasize that, if the spectrum of the Hamiltonian is real, the latter almost always² is in fact Hermitian when looking at it through proper glasses, i.e. when defining the norm in Hilbert space in a proper way. It was proved in [5] that the Hamiltonian with real non-degenerate spectrum must satisfy the properties (1) and (2). Then η defines the norm with respect to which the Hamiltonian H is Hermitian, while the Hamiltonian $\tilde{H} = O H O^{-1}$ is manifestly Hermitian with respect to the standard norm. In other words, the characterization ‘crypto-Hermitian’³ (Hermitian in disguise) reflects more adequately, in our opinion, the essence of the phenomenon, and we will stick to it in this paper⁴.

The modern history begins with the beautiful paper [9] (see also the recent review [10]), where this property was observed for a wide class of *PT*-symmetric polynomial potentials, like $V(x) = ix^3$.⁵ It was found to be discrete and real.

Since then, many crypto-Hermitian Hamiltonians have been discovered. We can mention the paper [11] where the spectrum of the Hamiltonian with hyperbolic and generalized hyperbolic complex *PT*-symmetric potentials was shown to be real in many cases. The simplest example of such a cryptoreal hyperbolic problem is the problem with the potential

$$V(x) = -\frac{V_1}{\cosh^2 x} + \frac{iV_2 \sinh x}{\cosh^2 x} \tag{3}$$

with $V_1 > 0$ and $|V_2| < V_1 + 1/4$.

In a recent paper [12], it was shown that apparently complex Hamiltonians obtained after the so-called *non-anticommutative* deformations [13] of certain supersymmetric quantum-mechanical and field theory models are in fact crypto-Hermitian and enjoy a real spectrum.

The problems with the potential $V(x) = e^{ix}$ or the potential (3) admit explicit analytic solutions. In [9], reality of the spectrum for the potentials $V(x) = x^2(ix)^\epsilon, \epsilon \geq 0$, was demonstrated explicitly by numerical solution of the corresponding Schrödinger equations

² ‘Almost’ means away from exceptional points [7] where the Hamiltonian involves Jordan blocks. We will discuss this issue later.

³ It was used first in [8].

⁴ Let us repeat for clarity: our *crypto-Hermiticity* means exactly the same as *quasi-Hermiticity* of [4] (a quasi-Hermitian Hamiltonian was defined there as the Hamiltonian that is Hermitian with respect to a generalized positive definite norm $\langle \psi | \eta \psi \rangle, \eta^\dagger = \eta$), the same as η -*Hermiticity* as defined in [6] and the same as *pseudo-Hermiticity* (1) with additional requirement (2).

⁵ A *PT*-symmetric potential $V(x)$ enjoys the property $V^*(-x) = V(x)$.

supplemented by semiclassical analysis. Later, a rigorous proof for the discreteness and reality of the spectrum in this problem was constructed [14]. In [15], it was shown that the Hamiltonians like

$$H = \frac{p^2 + x^2}{2} - g(ix)^{2n+1} \tag{4}$$

can be represented for small g in the forms (1) and (2). In other words, they can be obtained by a non-unitary transformation, $H = e^{-R} \tilde{H} e^R$, out of a manifestly Hermitian Hamiltonian \tilde{H} . The Hamiltonian \tilde{H} and the operator $R \equiv \ln O$ are calculated perturbatively as an infinite series in the coupling constant g .

In this paper, we suggest an approach capitalizing on a certain hidden gauge symmetry characteristic of crypto-Hermitian systems. The origin of this symmetry is very simple [16–18]. Consider a system with one dynamical degree of freedom. The classical Hamiltonian is a function $\mathcal{H}(p, x)$, which may be real or complex. Let us complexify the phase space variables,

$$\begin{aligned} x \rightarrow z = x + iy, \quad p \rightarrow \pi = p - iq, \\ \mathcal{H}(p, x) \rightarrow \mathcal{H}(\pi, z) = H(p, q; x, y) + iG(p, q; x, y), \end{aligned} \tag{5}$$

where H and G are real functions satisfying the Cauchy–Riemann relations

$$\frac{\partial H}{\partial p} + \frac{\partial G}{\partial q} = \frac{\partial H}{\partial y} + \frac{\partial G}{\partial x} = \frac{\partial H}{\partial q} - \frac{\partial G}{\partial p} = \frac{\partial H}{\partial x} - \frac{\partial G}{\partial y} = 0. \tag{6}$$

Two important properties follow:

- The function $H(p, q; x, y)$ can be treated as the Hamiltonian of a new system with double set of degrees of freedom. Indeed, the real and imaginary parts of the complexified equations of motion for the original system,

$$\dot{\pi} = -\partial \mathcal{H} / \partial z, \quad \dot{z} = \partial \mathcal{H} / \partial \pi,$$

coincide in virtue of (6) with the Hamilton equations of motion derived from $H(p, q; x, y)$.

- The Poisson bracket

$$\{H, G\}_{P.B.} = \frac{\partial H}{\partial x} \frac{\partial G}{\partial p} + \frac{\partial H}{\partial y} \frac{\partial G}{\partial q} - \frac{\partial H}{\partial p} \frac{\partial G}{\partial x} - \frac{\partial H}{\partial q} \frac{\partial G}{\partial y}$$

vanishes. This means that G is an integral of motion for the system described by H . The space of all classical trajectories is thus divided into classes characterized by a definite value of G . The class with $G = 0$ represents a particular interest. The condition $G = 0$ can be interpreted as a first class constraint and the dynamical system with the Hamiltonian H supplemented by the constraint $G = 0$ is a *gauge* system.

The plan of the paper is as follows. In the following section, we consider from this angle the simplest possible problem—the complexified oscillator. We note that this classical problem has at least three different quantum counterparts:

- (1) One can impose the analyticity constraint on the wavefunction, $\partial \psi / \partial \bar{z} = 0$ and solve the Schrödinger equation in the vicinity of the real axis. In this case, we reproduce the standard oscillator spectrum $E_n = 1/2 + n$. The same spectral problem is obtained when the gauge constraint is resolved at the classical level with the gauge choice $y = 0$.
- (2) One can impose the analyticity constraint and solve the Schrödinger equation in the vicinity of the imaginary axis. In this case, the spectrum $E_n = -1/2 - n$ involves negative energies and is bounded from above rather than from below. The same spectral problem is obtained when the gauge constraint is resolved at the classical level with the gauge choice $x = 0$.

- (3) Finally, one may not require analyticity, but rather impose, following Dirac, the gauge constraint $\hat{G}\Psi = 0$ on quantum states. In this case, the spectrum is $E_n = n$, where n can be positive, zero or negative. Still, the quantum problem is well defined, and the evolution operator is unitary.

In section 3, we consider the classical dynamics of the Hamiltonian

$$H = \frac{\pi^2}{2} - (iz)^{2n+1}. \tag{7}$$

We find different sets of trajectories with positive, and also with *negative* energies. Section 4 is devoted to the quantum dynamics of (7) and of the mixed Hamiltonian (4), with the analyticity constraint imposed on wavefunctions. The complex plane of z is divided then into several regions. In some of them the spectrum is discrete, while in others it is continuous or empty. For $n = 1$, we reproduce the results of [9]. For $n > 1$, one can formulate n different spectral problems with discrete positive definite spectrum formulated in the different regions of the complex plane⁶. When $g \rightarrow 0$, the spectrum of the mixed Hamiltonian approaches the oscillator spectrum, but for the problems formulated in the sectors not comprising real axis, the transformation pattern is very nontrivial involving an infinite set of ‘phase transitions’ in the coupling. At each of such ‘phase transition’ (or exceptional [7] point $g_*^{(j)}$), a pair of eigenstates of the mixed Hamiltonian coalesce such that at this very point the Hamiltonian involves a nondiagonalizable Jordan block. Beyond this point ($g < g_*^{(j)}$), a pair of complex conjugate eigenvalues should appear. In other words, the Hamiltonian (4) is cryptoreal in this situation only for large enough g (and, obviously, it is Hermitian for $g = 0$). The last section is devoted as usual to discussions. In particular, we discuss the Dirac spectral problem for the Hamiltonian (7) when the gauge constraint is imposed on the wavefunctions as an operator condition. This problem has no analytic solution and is difficult to resolve numerically. Still, based on semiclassical reasoning, we argue that, similar to what we had in the oscillator case, the spectrum there might be discrete and unbounded both from below and above. We also point out the similarity of this problem to some others that we have previously analyzed, which are described by higher-derivative Lagrangians and involve ghosts. We speculate that, in spite of their presence, unitarity is not violated.

2. Complex oscillator

Consider the complex Hamiltonian

$$\mathcal{H}(\pi, z) = \frac{\pi^2 + z^2}{2}. \tag{8}$$

Its real and imaginary parts are

$$H = \frac{p^2 + x^2}{2} - \frac{q^2 + y^2}{2} \tag{9}$$

and

$$G = -pq + xy. \tag{10}$$

Consider the classical dynamics of H . The classical trajectories are

$$x = A \sin(t + \phi_1), \quad p = A \cos(t + \phi_1), \quad y = B \sin(t + \phi_2), \quad q = -B \cos(t + \phi_2). \tag{11}$$

⁶ That was observed in [19]. The proof of reality and discreteness of the spectrum for all *symmetric* (see below) spectral problems was given in [20].

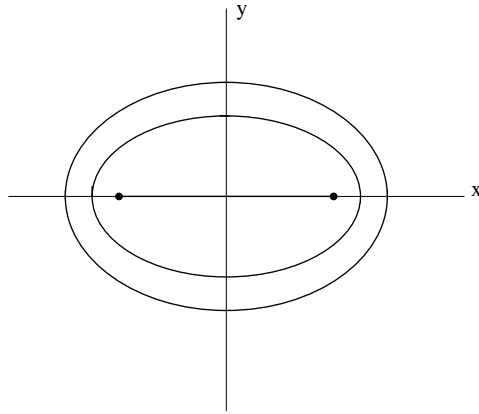


Figure 1. Family of oscillator trajectories with the same positive energy. For negative energies, the picture is rotated by $\pi/2$.

Generically, they have complex energies. If we require the energies to be real, i.e. impose the constraint $G = 0$, the relation

$$AB \cos(\phi_1 - \phi_2) = 0 \tag{12}$$

follows. For each value of the energy, positive or negative, there is a set of trajectories (cofocal ellipses) with the same period (see figure 1).

In the case under consideration, the period is the same for *all* energies, but this is the specifics of oscillator. The fact that the period is the same for all trajectories of a *given* energy has, however, a general nature. In fact, it is a consequence of the gauge symmetry of the problem.

The latter is simply the symmetry generated by the constraint G . Infinitesimally [18],

$$\begin{aligned} \delta_G x &= -\alpha \{G, x\}_{P.B.} = -\alpha q, & \delta_G y &= -\alpha \{G, y\}_{P.B.} = -\alpha p, \\ \delta_G p &= -\alpha \{G, p\}_{P.B.} = -\alpha y, & \delta_G q &= -\alpha \{G, q\}_{P.B.} = -\alpha x. \end{aligned} \tag{13}$$

This is a phase space symmetry. To represent it as a conventional gauge symmetry acting only on the coordinates, one should introduce the Lagrange multiplier $\lambda(t)$ and write the canonical Lagrangian as

$$L = p\dot{x} + q\dot{y} - H - \lambda G, \tag{14}$$

Expressing out the momenta,

$$p = \frac{\dot{x} - \lambda \dot{y}}{1 + \lambda^2}, \quad q = -\frac{\dot{y} + \lambda \dot{x}}{1 + \lambda^2}, \tag{15}$$

we obtain

$$L = \frac{\dot{x}^2 - \dot{y}^2 - 2\lambda \dot{x} \dot{y}}{2(1 + \lambda^2)} + \frac{y^2 - x^2}{2} - \lambda xy. \tag{16}$$

The gauge transformations amount to shifting the Lagrange multiplier λ by (a derivative of) an arbitrary function of time $\dot{\alpha}(t)$, supplemented by the transformations of dynamic variables

x, y generated by the constraint G .

$$\begin{aligned} \delta_{\text{gauge}}x &= -\alpha q = \frac{\alpha(\dot{y} + \lambda\dot{x})}{1 + \lambda^2}, \\ \delta_{\text{gauge}}y &= -\alpha p = -\frac{\alpha(\dot{x} - \lambda\dot{y})}{1 + \lambda^2}, \\ \delta_{\text{gauge}}\lambda &= \dot{\alpha}. \end{aligned} \tag{17}$$

Indeed, one can explicitly verify that the Lagrangian (16) is invariant, up to a total derivative, with respect to the transformations (17).

The transformations δx and δy in equations (13) and (17) have a clear meaning. Any Hamiltonian system is invariant with respect to time translations $t \rightarrow t - a$ that transform a solution $z(t) \rightarrow z(t - a)$. Their generator is the Hamiltonian H . In our case, however, besides $H \equiv \text{Re}(\mathcal{H})$, we have another integral of motion $G \equiv \text{Im}(\mathcal{H})$. It generates a shift of time by an *imaginary* amount, $t \rightarrow t - i\alpha$ and transforms $z(t) \rightarrow z(t - i\alpha)$. Infinitesimally, this coincides with equation (17) (with partial gauge fixing $\lambda = 0$).

The shift $z(t) \rightarrow z(t - a)$ is the shift along the trajectory, leaving it unchanged. But the shift $z(t) \rightarrow z(t - i\alpha)$ transforms one trajectory into another. It is this shift which relates different ellipses in figure 1 (it is straightforward to check by substituting for $z(t)$ the exact analytic solution (11) with $\phi_1 - \phi_2 = \pi/2$). Such families of closed trajectories of a given energy and the same period (obviously, if $z(t)$ is periodic, $z(t - i\alpha)$ is also periodic with the same real period) exist also for more complicated cases. We will discuss it in the following section.

Let us go over to quantum dynamics. There are two basic ways to quantize gauge systems⁷: (i) by explicitly resolving the constraints and quantizing the Hamiltonian with a reduced number of degrees of freedom and (ii) by not resolving the constraints classically, but rather solving the system

$$\hat{H}\Psi = E\Psi, \quad \hat{G}\Psi = 0. \tag{18}$$

We will see that, in the case under consideration, these two approaches are not quite equivalent, in contrast to what is usually assumed!

- Let us first try to resolve the constraint $G = 0$ classically. This can be done by fixing the gauge, i.e. by imposing the additional constraint $\chi(p, q; x, y) = 0$, where $\{G, \chi\}_{P.B.} \neq 0$ (so that the primary constraint $G = 0$ and the gauge fixing constraint $\chi = 0$ are independent). Resolving the system $G = \chi = 0$, we are left with a reduced number of dynamical variables. Generically, their number is equal to the number of initial degrees of freedom minus the number of primary constraints. In our case, $N_{\text{reduced}} = 2 - 1 = 1$. One can, for example, choose $\chi = y = 0$. The reduced Hamiltonian system will in this case be just $H^* = (p^2 + x^2)/2$ with the spectrum $E_n = 1/2 + n$. On the other hand, if choosing the gauge $\chi = x = 0$, the reduced Hamiltonian is $H^* = -(q^2 + y^2)/2$ with a different spectrum $E_n = -1/2 - n$. In other words, there are two essentially *different* gauge choices leading to *different* reduced Hamiltonians. One can obtain either oscillator with positive energies, or oscillator with negative energies, but not both.

To understand what happened, look again at the trajectories in figure 1. They represent, as we have seen, gauge copies of one another. The gauge fixing procedure should pick out one of these copies, while getting rid of all others. And, indeed, the condition $y = 0$ does this job by pinpointing the trajectory going along the real axis. However, none of these trajectories are compatible with the condition $x = 0$. On the other hand, for the

⁷ This problem was first posed and resolved by Dirac and is treated pedagogically in many books. See, e.g. [21].

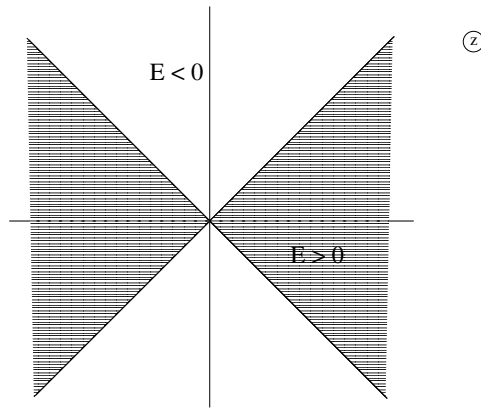


Figure 2. Sectors with positive and negative energies in the spectrum.

family of the trajectories with negative energies, one can impose $x = 0$ (and pinpoint the trajectory going along the imaginary axis), but not $y = 0$.⁸

Two spectral problems with positive and negative energies can alternatively be defined using the approach of [9]. To this end, one should require that the wavefunction represents an analytic function of $z = x + iy$. The spectrum $E_n = 1/2 + n$ is then realized by the standard oscillator functions continued analytically to complex arguments. For example, the wavefunction of the ground state is $\exp(-z^2/2)$. It falls down exponentially on the real axis and also on the lines $z = us, s \in (-\infty, \infty), |\text{Arg}(u)| < \pi/4$. The spectrum $E_n = -1/2 - n$ is realized by the functions like $\exp(z^2/2)$ that fall down exponentially along the imaginary axis and in the sector $|\text{Arg}(z)| > \pi/4$ (see figure 2).

The lines $\text{Arg}(z) = \pm\pi/4$ are closely related with the Stokes lines of the oscillator Schrödinger equation. The Stokes lines are defined [22] as the lines that pass through turning points and satisfy the condition

$$\text{Im} \left(\int_{z_0}^z \pi(w) dw \right) = \left(\int_{z_0}^z \sqrt{2(E - w^2)} dw \right) = 0 \tag{19}$$

(z_0 is the position of the turning point). The asymptotes of Stokes lines at large values of $|z|$ are the straight lines separating the sectors in figure 2. When crossing a Stokes line, the asymptotics of the solution to the differential equation changes its nature.

- Another approach is to solve the system (18). Were the constraint $\hat{G}\Psi = 0$ not imposed, the spectrum would be $E_{mn} = n - m$ with the eigenfunctions $|nm\rangle = |n\rangle_x |m\rangle_y$. It is infinitely degenerate at each level. The constraint $G = 0$ picks up only one representative of the set of eigenstates of H with a given energy. For example, the zero-energy state annihilated by G is

$$\Psi_0 = \sum_{k=0}^{\infty} |2k, 2k\rangle (-1)^k \frac{(2k - 1)!!}{(2k)!!}. \tag{20}$$

At large k , the coefficient is proportional to $1/\sqrt{k}$, i.e. the normalization integral for (20) diverges logarithmically.

⁸ The trajectories in figure 1 are related by gauge transformations with constant α . But one can easily prove that one cannot obtain a configuration with $x(t) = 0$ out of a configuration with $y(t) = 0$ by a generic gauge transformation (17). Indeed, the energy functional is positive definite when $y(t) = 0$ and negative definite when $x(t) = 0$.

Similarly, only one eigenstate is left at each energy level. The full spectrum is discrete,

$$E_n = n \tag{21}$$

with positive, negative or zero integer n . It is unbounded both from below and above. This notwithstanding, the spectral problem is well defined and the evolution operator

$$\mathcal{K}(x', x) = \sum_n \Psi_n^*(x') \Psi_n(x) e^{int} \tag{22}$$

is unitary⁹.

Comparing the results we obtained under two quantization procedures, one can make two observations. First, the spectrum is shifted by $1/2$. The ambiguity whether $E_n = n$ or $E_n = n + 1/2$ has the same nature as the well-known ordering ambiguity—there are many different quantum problems having the same classical limit. The second observation is that, on top of the ordering ambiguity, there is in this case also another ambiguity associated with gauge choice. With any gauge choice, half of the spectrum involving either the states with negative or with positive energies is lost.

A lesson that can be drawn from this simple toy model is that, for gauge systems, fixing the gauge classically and quantizing afterward may be dangerous. Certain essential features of the spectral problem (18) may be lost.

3. The potential $-(ix)^{2n+1}$: classical dynamics

Having being equipped with necessary tools, we may proceed now with the analysis of the Hamiltonians of interest written in equations (4) and (7). We will concentrate mainly on the Hamiltonian (7) without the oscillator term in the potential.

Let first $n = 1$. Consider the complex Hamiltonian

$$\mathcal{H} = \frac{\pi^2}{2} + iz^3 \tag{23}$$

with $z = x + iy$, $\pi = p - iq$. Its real and imaginary parts are, respectively,

$$H = \frac{p^2 - q^2}{2} + y^3 - 3yx^2, \quad G = -pq + x^3 - 3xy^2. \tag{24}$$

Consider the dynamics of the system described by the Hamiltonian H and the constraint G . It can be treated as a gauge system. The equations of motion follow from the Hamiltonian $H + \lambda G$, where λ is the Lagrange multiplier. They have the form

$$\begin{aligned} \dot{p} &= 6xy + 3\lambda(y^2 - x^2), & \dot{x} &= p - \lambda q, & \dot{q} &= 3(x^2 - y^2) + 6\lambda xy, \\ \dot{y} &= -q - \lambda p, & G &= -pq + x^3 - 3xy^2 = 0. \end{aligned} \tag{25}$$

The Lagrangian (14) is invariant up to a total derivative with respect to gauge transformations (17) with time-dependent parameter $\alpha(t)$. To find the classical solutions, we need first to fix the gauge. A convenient *partial* gauge fixing corresponds to the condition $\lambda(t) = 0$, in which case the equations are reduced to

$$\dot{p} = 6xy, \quad \dot{x} = p, \quad \dot{q} = 3(x^2 - y^2), \quad \dot{y} = -q, \quad G = 0. \tag{26}$$

The solutions to (26) belong to two classes. (1) Runaway trajectories, which reach infinity at finite time. These are, for example, the trajectories with initial conditions $x(0) = \dot{x}(0) = 0$.

⁹ See [23] for detailed discussion of this and related issues.

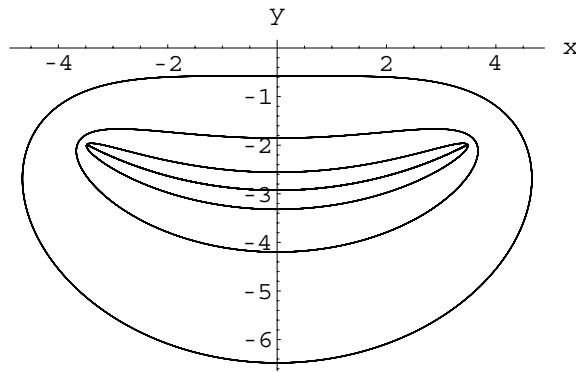


Figure 3. Family of trajectories with $E = 64$.

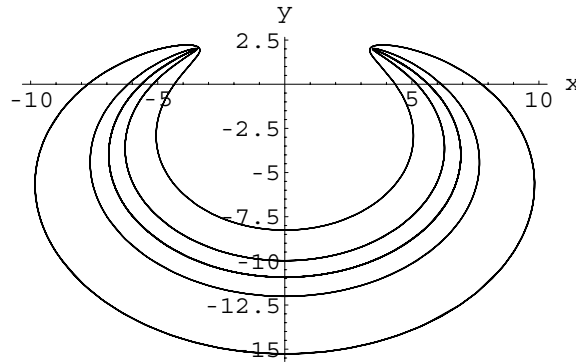


Figure 4. Family of trajectories with $E = -64$.

They run away in the positive y directions. (2) Besides, there are families of closed orbits related to each other by gauge transformations (17) with constant α . For *positive* energies, these families, depicted in figure 3, were found in [19]. This family has one distinguished member (one can call it a *stem* trajectory): the trajectory which connects the turning points (the points where the momenta p, q vanish) with the coordinates.

$$y_* = -\frac{E^{1/3}}{2}, \quad x_* = \pm \frac{\sqrt{3}E^{1/3}}{2} \tag{27}$$

(there is also the turning point $x = 0, y = E^{1/3}$, but the trajectories originating there run away rather than coming back).

Note that the families of trajectories with negative energies also exist (see figure 4). They stem from the trajectories connecting the turning points

$$y_{**} = \frac{(-E)^{1/3}}{2}, \quad x_{**} = \pm \frac{\sqrt{3}(-E)^{1/3}}{2}. \tag{28}$$

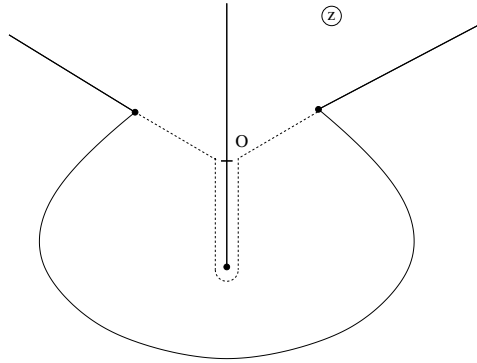


Figure 5. Analytic structure of $\pi_{cl}(z)$ for negative energies. (Solid line) original contour, (dashed line) deformed contour and (thick solid lines) the cuts.

Let us calculate for future purposes the action on these trajectories. Using the fact that the action for all orbits belonging to one family is the same, one can write

$$S = \oint (p dx + q dy) = 2 \operatorname{Re} \int_{z_1}^{z_2} \pi dz = 2 \operatorname{Re} \int_{z_1}^{z_2} \sqrt{2(E - iz^3)} dz, \quad (29)$$

where $z_{1,2}$ are the turning points. For the trajectories of positive energies, the integral can be easily done by deforming the contour such that it passes the origin¹⁰,

$$S_+ = 4 \operatorname{Re} \int_0^{z_*} \sqrt{2(E - iz^3)} dz = 4 \operatorname{Re}(z_*) \sqrt{2E} \int_0^1 \sqrt{1 - s^3} ds = \sqrt{6\pi} E^{5/6} \frac{\Gamma(4/3)}{\Gamma(11/6)}. \quad (30)$$

To calculate the action for negative-energy orbits, one has to take into account the fact that the turning points are at the same time the branching points of the integrand in (29). For positive energies, the corresponding cuts do not hinder the deformation of the contour, but, for negative energies, they do. The cuts should be drawn such that the original path does not cross them. The deformed contour also should avoid crossing the cuts. The corresponding structure of the cuts, the original and deformed contours are shown in figure 5.

It is clear from the figure that the deformed contour involves four pieces: (i) from the left turning point to the origin, (ii) and (iii) from the origin down the cut and up again and (iv) from the origin to the right turning point. A simple analysis shows that the contribution of the parts (i)–(iv) involves an extra factor $\sin(\pi/6) = 1/2$ compared to the contribution of the parts (ii) and (iii). All together, the integral for S_- involves an extra factor $[1 + \sin(\pi/6)]/\cos(\pi/6) = \sqrt{3}$ compared to the integral (30) for S_+ with the same absolute value of energy. In other words,

$$S_- = 3\sqrt{2\pi} (-E)^{5/6} \frac{\Gamma(4/3)}{\Gamma(11/6)}. \quad (31)$$

Consider now the complex Hamiltonian¹¹

$$\mathcal{H} = \frac{\pi^2}{2} - iz^5 \quad (32)$$

¹⁰ This result (in somewhat different normalization) was obtained in [19].

¹¹ The sign of the potential corresponds to the convention (7) and to the conventions of [9, 19]. These conventions are convenient to make the physics of the systems (7) with different n more similar.

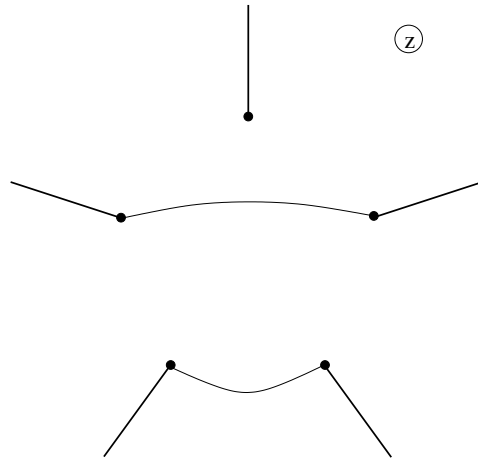


Figure 6. Turning points, cuts and stem trajectories for the potential $-iz^5$. Positive energies.

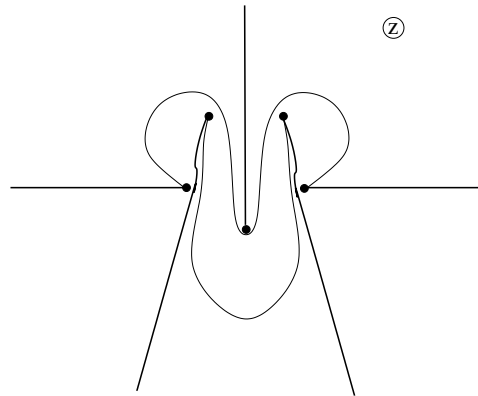


Figure 7. Turning points, cuts and stem trajectories for the potential $-iz^5$. Negative energies. The cuts are drawn not to interfere with the paths.

with real and imaginary parts

$$\begin{aligned}
 H &= \frac{p^2 - q^2}{2} + y^5 - 10y^3x^2 + 5yx^4, \\
 G &= -pq - x^5 + 10y^2x^3 - 5xy^4 \rightarrow 0.
 \end{aligned}
 \tag{33}$$

Again, there are runaway trajectories taking a finite time to reach infinity in the positive y direction. Besides, there are *four* families of closed orbits: two families with positive energies and two families with negative energies. The structure of the turning points, associated cuts and the stem trajectories connecting the turning points are shown schematically in figures 6 and 7 for positive and negative energies, respectively. Let us find the classical action at these trajectories. For positive energies,

$$\begin{aligned}
 S_+^{\text{up}} &= 4 \cos \frac{\pi}{10} \sqrt{2} \int_0^1 \sqrt{1 - s^5} ds E^{7/10} = 2\sqrt{2\pi} \cos \frac{\pi}{10} \frac{\Gamma(6/5)}{\Gamma(17/10)} E^{7/10}, \\
 S_+^{\text{down}} &= 4 \cos \frac{3\pi}{10} \sqrt{2} \int_0^1 \sqrt{1 - s^5} ds E^{7/10} = 2\sqrt{2\pi} \cos \frac{3\pi}{10} \frac{\Gamma(6/5)}{\Gamma(17/10)} E^{7/10}.
 \end{aligned}
 \tag{34}$$

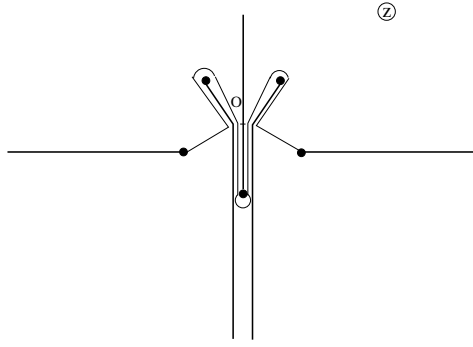


Figure 8. The deformed upper contour wiggling around the deformed cuts.

For negative energies,

$$\begin{aligned}
 S_-^{\text{up}} &= 2\sqrt{2\pi} \left(1 + 2 \sin \frac{3\pi}{10} + \sin \frac{\pi}{10} \right) \frac{\Gamma(6/5)}{\Gamma(17/10)} (-E)^{7/10}, \\
 S_-^{\text{down}} &= 2\sqrt{2\pi} \left(1 + \sin \frac{3\pi}{10} \right) \frac{\Gamma(6/5)}{\Gamma(17/10)} (-E)^{7/10}.
 \end{aligned}
 \tag{35}$$

The superscript ‘up’ in equation (35) refers to the upper trajectory in figure 7 going between the points $e^{11i\pi/10}$ and $e^{-i\pi/10}$. The result for S_-^{down} is obtained in the same way as the result (31), with the factor $1 + \sin(\pi/6)$ being replaced by $1 + \sin(3\pi/10)$. When deforming the contour for the upper trajectory, we find, in addition to the parts composing the deformed contour of the lower trajectory and giving the factor $1 + \sin(3\pi/10)$, also two extra pieces with the contribution $\sim \sin(3\pi/10) + \sin(\pi/10)$. The origin of all these factors can be clearly seen, if deforming the contour and the cuts in the way shown in figure 8. All the pieces (of nonzero length) connect the branching points to the center of the pentagon $z = 0$.

By the same token, for the potential $-(iz)^{2n+1}$, there are $2n$ families of the trajectories: n families with positive energies and n families with negative energies.

As we have seen, the classical dynamics of the system with the potential $-(iz)^{2n+1}$ is similar in many respects to the complex oscillator dynamics: a distinct feature of both systems are the families of closed orbits with positive and negative energies, the members of one family being interrelated by gauge transformations. There are also two important differences. First, the system $-(iz)^{2n+1}$ involves besides closed orbits also singular runaway trajectories. Second, for the complex oscillator, the stem trajectories for the families of orbits could be conveniently obtained by fixing the gauge $y = 0$ or $x = 0$. But for the system $-(iz)^{2n+1}$, this is not true. To begin with, the stem trajectories displayed above are essentially complex. This observation is not yet sufficient, however, because it does not exclude a conceivable in principle possibility that the trajectories can be put onto the real (or imaginary) axis by a complicated gauge transformation (17) with nontrivial $\alpha(t)$.

Let us find out what happens if we *do* fix the gauge $y = 0$ for the system (24) and (25). From $G = 0$, we deduce $q = x^3/p$ and hence the Hamiltonian is reduced to

$$H^* = \frac{p^2}{2} - \frac{x^6}{2p^2}.
 \tag{36}$$

The corresponding equations of motion

$$\dot{x} = p + \frac{x^6}{p^3}, \quad \dot{p} = \frac{3x^5}{p^2}
 \tag{37}$$

follow from (25) with $\lambda = -x^3/p^2$. We see now that the reduced Hamiltonian (36) is neither positive nor negative definite and involves *only* runaway trajectories. Closed orbits have disappeared! This is another manifestation of the fact discussed in the previous section that fixing the gauge at the classical level is not an innocent procedure and may lead to a loss of important dynamic features. For the complex oscillator with the gauge choice $y = 0$, half of the orbits (the orbits with negative energies) were lost. For the system $-(iz)^{2n+1}$, *all* closed orbits are lost and we are left only with runaway solutions.

Let us discuss the relationship of the Hamiltonian (36) to another Hamiltonian obtained from (23) by a non-unitary rotation technique in the spirit of [15]. Let us multiply the potential by a coupling constant g , $ix^3 \rightarrow igx^3$, and find an operator R such that the rotated Hamiltonian $\tilde{H} = e^R(p^2/2 + igx^3)e^{-R}$ be manifestly real. Then, R can be presented as an infinite series over the coupling constant,

$$R = -\frac{gx^4}{4p} + O(g^3)$$

and [24] (see also section V of [25])

$$\tilde{H} = \frac{p^2}{2} + \frac{3g^2x^6}{8p^2} + O(g^4). \quad (38)$$

We see that the H^* and \tilde{H} have similar structure, but the coefficients differ. This does not represent a paradox because \tilde{H} , in contrast to H^* , involves the whole infinite series in g . Anyway, all the terms in this series are nonlocal, and one cannot obtain from this, say, the spectrum of quantum Hamiltonian as a perturbative series in g . The non-unitary rotation technique is better suited to the problems like (4), where all the terms in the perturbative series for \tilde{H} are local.

Coming back to fixing the gauge with the condition $y = 0$, it does not work well also for the mixed system (4), however small g is. The extra piece in H^* is still nonlocal and singular at the turning point of the unperturbed oscillator trajectory where the momentum p vanishes. As a result, the trajectory does not *turn* there, but rather stumbles and runs away.

4. Quantum dynamics

Let us discuss now quantum dynamics of the Hamiltonians (23) and (32). Consider equation (23) first. In section 2, we outlined two regular ways to quantize gauge systems: (i) resolving the constraint(s) at the classical level and quantizing afterward and (ii) solving the system of differential equations (18) with proper boundary conditions.

To resolve the constraints classically, one has to fix the gauge. Unfortunately, as we have just seen, it is difficult to find a clever way to do it in our case. A natural gauge fixing leads to the problem involving *only* runaway trajectories. This means trouble and, indeed, for the highly nonlocal and not positive definite Hamiltonian (36), one cannot formulate a well-defined quantum problem with a unitary evolution operator.

Another approach is to solve the system (18). This is a nontrivial numerical problem. Indeed, one-dimensional spectral problems can be easily solved with Mathematica, but in this case the problem is essentially two-dimensional, which is much trickier. What is even more important, the operators H and G in (18) are not elliptic, as usual, but hyperbolic. It is not thus evident that a reasonable solution to this problem exists. We will discuss this question somewhat more in the last section, but, basically, we leave it for future studies.

Table 1. Exact and semiclassical spectra for the potential ix^3 .

k	0	1	2	3
E_{exact}^k	0.763	2.711	4.989	7.465
E_{semicl}^k	0.722	2.698	4.980	7.458

There is, however, a way to define a consistent spectral problem related to the Hamiltonian (23) [9]. Forget for a moment all what was said above about complexification and consider the Schrödinger equation at the real axis,

$$\left[-\frac{1}{2} \frac{\partial}{\partial x^2} + ix^3 \right] \Psi = E\Psi, \tag{39}$$

with the condition that the wavefunction falls down at $x = \pm\infty$. It is convenient to pose the problem not on the whole line $(-\infty, \infty)$, but on the half-line $(0, \infty)$. One can do it by exploiting the PT symmetry of the potential (the property $V(-x) = V^*(x)$). It dictates that for any solution $\Psi(x)$ of equation (39), the function $\Psi^*(-x)$ is also the solution with the same eigenvalue. The functions

$$\Psi_{\pm}(x) = \Psi(x) \pm \Psi^*(-x) \tag{40}$$

with the symmetry properties $\Psi_+(-x) = \Psi_+^*(x)$ and $\Psi_-(-x) = -\Psi_-^*(x)$ also satisfy this equation. We are hence allowed to consider the equations for PT -even function $\Psi_+(x)$ and PT -odd function $\Psi_-(x)$ separately. In this case (in contrast, e.g. to the standard oscillator problem), the equation for $\Psi_-(x)$ does not give anything new. Indeed, one can make a PT -odd function out of a PT -even one by simply multiplying the latter by i . A generic solution to (39) is obtained by multiplying a PT -even solution by an arbitrary complex factor.

The condition $\Psi(-x) = \Psi^*(x)$ means that $\Psi(0)$ is real while $\Psi'(0)$ is imaginary. By turning the computer on, everybody can be convinced that equation (39) with the boundary conditions

$$\text{Re} \left(\frac{\Psi'(0)}{\Psi(0)} \right) = 0, \quad \Psi(\infty) = 0 \tag{41}$$

has solutions at real positive discrete values of E . The remarkable fact is that these values are very close to *semiclassical* energies associated with the family of the closed orbits in figure 3 obtained from the quantization condition

$$S(E_k) = \pi(2k + 1), \tag{42}$$

with the function $S(E)$ being given by equation (30). When $k \rightarrow \infty$, the spectral values extracted from equations (39) and (41) and the semiclassical values extracted from equations (30) and (42) rapidly converge. The exact and semiclassical values for E_k for first few levels [9] are shown in table 1.

Once the solution is obtained, one need not stay on the real axis. Actually, the solution can be continued analytically to complex values of the argument z in the regions

$$\left| \text{Arg}(z) + \frac{\pi}{10} \right| \leq \frac{\pi}{5}, \quad \left| \text{Arg}(z) - \frac{11\pi}{10} \right| \leq \frac{\pi}{5}. \tag{43}$$

In other words, the spectral problem

$$\left[-\frac{1}{2\Phi^2} \frac{\partial}{\partial s^2} + is^3\Phi^3 \right] \Psi = E\Psi, \quad \text{Re} \left(\frac{\partial\Psi/\partial s}{\Phi\Psi} \right) \Big|_{s=0} = 0, \quad \Psi(\infty) = 0, \tag{44}$$

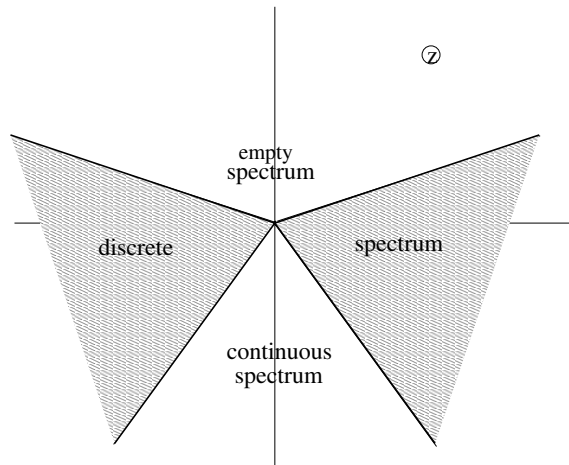


Figure 9. Spectral problem (44) in the complex z plane.

with $\Phi = e^{i\alpha}$, still has a solution when α lies within the interval (43), and the spectral values are exactly the same as for the problems (39) and (41). When

$$-7\pi/10 < \alpha < -3\pi/10, \tag{45}$$

the spectrum is continuous: any positive or negative energy is acceptable. This is especially clearly seen for $\alpha = -\pi/2$ (meaning $\Phi = -i$). The problem (44) is then reduced to

$$\left[-\frac{1}{2} \frac{\partial}{\partial s^2} + s^3 \right] \Psi = -E\Psi, \tag{46}$$

$$\Psi(0) = 1, \quad \Psi(\infty) = 0, \quad \text{Im}[\Psi'(0)] = 0.$$

The real part of $\Psi'(0)$ is not fixed, however, and tuning this parameter, one can obtain the solution dying at infinity at any energy¹². A numerical analysis shows that it is true in the whole interval (45).

On the other hand, for $\pi/10 < \alpha < 9\pi/10$ the problem (44) has no solution whatsoever: the spectrum is empty¹³. This is all illustrated in figure 9.

The system of the lines separating the sectors in figure 9 form together with the positive imaginary axis the system of the asymptotes of the Stokes lines of the Schrödinger equation with the potential iz^3 . Generically, for a polynomial potential of order n , such system involves $n + 2$ lines forming equal angles $2\pi/(n + 2)$ [22].

The spectral problem (44) corresponds to the family of the classical orbits in figure 3 with *positive* energies. As we have seen (in figure 4), there are also orbits with negative energies. Using the result (31), it is not difficult to find the corresponding semiclassical energies,

$$E_k = - \left[\frac{(2k + 1)\Gamma(11/6)}{\Gamma(4/3)} \right]^{6/5} \left(\frac{\pi}{18} \right)^{3/5}. \tag{47}$$

¹² By modifying the spectral problem by, for example, imposing the conditions $\Psi(0) = \Psi(\infty) = 0$ instead of (41), one can force the spectrum to be discrete and negative definite. But the condition $\Psi(0) = 0$ is artificial and has no physical motivation. In particular, the discrete negative definite spectrum thus obtained has nothing to do with the semiclassical spectrum (47).

¹³ If lifting the requirement that the wavefunction dies away at infinity, the spectrum would again become continuous.

Table 2. Exact and semiclassical spectra for the potential $-ix^5$.

k	0	1	2	3
E_{exact}^k	0.710	2.660	5.458	8.788
E_{semicl}^k	0.543	2.608	5.410	8.750

One may suggest that a spectral problem should exist for which equation (47) would represent a semiclassical approximation. However, no such problem is known¹⁴. At least, it is not known in the standard form of boundary problem for some differential operator. One still can calculate the ‘exact spectrum’ of such nonexistent (or very well hidden) problem by calculating corrections to the result (47) and representing E_{exact}^k as a series in semiclassical parameter $\sim 1/S_{\text{cl}}$. As this series is probably asymptotic, this method gives an intrinsic uncertainty in the spectrum $\sim \exp\{-CS_{\text{cl}}\}$. However, the closeness of exact energies of positive energy states and their semiclassical approximations (see table 1) and the calculations of higher-order corrections in [19] suggest that this uncertainty is not large even for the ‘sky state’ in equation (47) with $k = 0$ and $S_{\text{cl}} = \pi$. It rapidly decreases with increase of k .

Consider now the Hamiltonian (32). Again, one can solve the Schrödinger equation with the potential $-ix^5$ at the real axis with boundary conditions $\Psi(\pm\infty) = 0$ and find a discrete spectrum with real positive energies. As is seen from table 2, these exact energies are very close to the semiclassical values determined from the quantization condition

$$S_+^{\text{up}} = \pi(2k + 1), \tag{48}$$

where S_+^{up} given in equation (34) is evaluated for the *upper* trajectory in figure 6.

As we see, the semiclassical approximation works somewhat worse in this case than for the potential ix^3 . But it works.

We can now leave the real axis and solve the spectral problem

$$\begin{aligned} \left[-\frac{1}{2\Phi^2} \frac{\partial}{\partial s^2} - is^5\Phi^5 \right] \Psi &= E\Psi, \\ \text{Re} \left(\frac{\partial \Psi / \partial s}{\Phi \Psi} \right) \Big|_{s=0} &= 0, \quad \Psi(\infty) = 0, \end{aligned} \tag{49}$$

with $\Phi = e^{i\alpha}$. The solution *with the same spectrum* exists for

$$\left| \alpha - \frac{\pi}{14} \right| \leq \frac{\pi}{7} \quad \text{or} \quad \left| \alpha - \frac{13\pi}{14} \right| \leq \frac{\pi}{7}. \tag{50}$$

For

$$\left| \alpha + \frac{3\pi}{14} \right| \leq \frac{\pi}{7} \quad \text{or} \quad \left| \alpha - \frac{17\pi}{14} \right| \leq \frac{\pi}{7}, \tag{51}$$

the solution still exists, but the spectrum is different. Its semiclassical approximation comes not from the quantization condition (48), but rather from the quantization condition

$$S_+^{\text{down}} = \pi(2k + 1) \tag{52}$$

derived for the *lower* stem trajectory in figure 6. The exact and semiclassical energy values for this case are given in table 3.

¹⁴ And here is an important difference with the complex oscillator problem discussed in section 2, where the spectral problem with the spectrum $E_k = -k - 1/2$ was perfectly well defined.

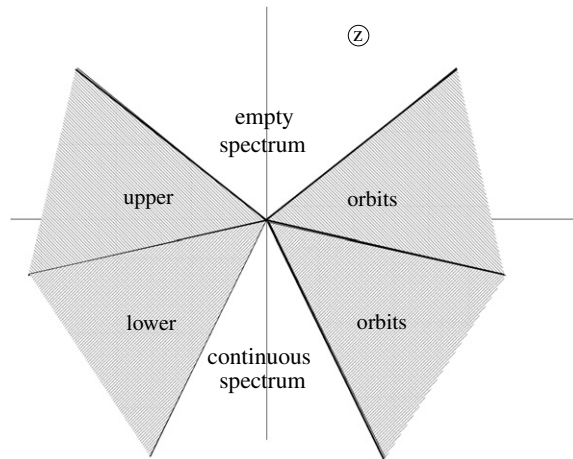


Figure 10. Spectral problem (49) in the complex z plane.

Table 3. Exact and semiclassical spectra for the potential $-iz^5$ in the region (51).

k	0	1	2	3
E_{exact}^k	1.163	5.234	10.795	17.428
E_{semicl}^k	1.080	5.186	10.759	17.400

Finally, for $-9\pi/(14) < \alpha < -5\pi/(14)$, the spectrum is continuous while, for $3\pi/(14) < \alpha < 11\pi/(14)$, the spectrum is empty. The corresponding regions in the complex z plane are displayed in figure 10.

For the Hamiltonian (4), there are n different nontrivial spectral problems with discrete spectrum defined in the sectors

$$\left| \alpha + \frac{(2n-1)\pi}{2(2n+3)} - \frac{2\pi m}{2n+3} \right| \leq \frac{\pi}{2n+3} \quad \text{or mirror image,} \tag{53}$$

$m = 0, \dots, n-1$.¹⁵ They correspond to n different families of classical orbits with positive energies for the potential $-(iz)^{2n+1}$. The problem studied in details in [9, 19] was defined in the sector $m = 0$.¹⁶ We concentrate in this paper on imaginary potentials like in equation (7). But one can equally well [9, 19] consider the potentials

$$V(z) = -(iz)^{2n}. \tag{54}$$

For the quartic potential $\sim -z^4$, there are four turning points, two sets of symmetric classical positive energy orbits and a corresponding spectral problem defined in the sector $|\alpha - \pi/6| \leq \pi/6$ and its mirror images. For the potential $\sim z^6$ we have, besides the standard spectral problem on the real axis, also a nontrivial problem in the sector $|\alpha - \pi/4| \leq \pi/8$, etc. For generic n , the potential (54) admits $n/2$ different spectral problems when n is even and $(n+1)/2$ different spectral problems when n is odd.

The presence of several different quantum problems associated with a given classical potential seems to be natural in view of our analysis for the complex oscillator, where two

¹⁵ There are also asymmetric spectral problems. One can, for example, go from infinity to zero along the line $\alpha = 17\pi/14$ and to infinity from zero along the line $\alpha = \pi/14$. But such problems have complex eigenvalues [20] and we are not considering them.

¹⁶ In [19], the problem with $m = 1$ was also considered. It was represented as the problem with the potential $V(x) = x^4(ix)^\epsilon$. But the results for the spectrum were given there only for negative ϵ .

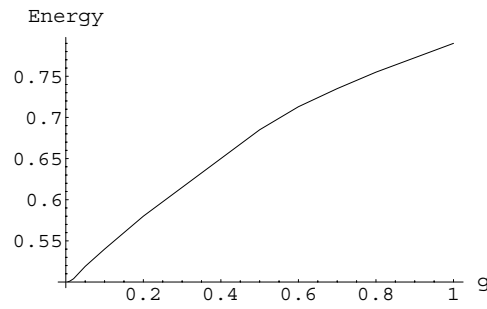


Figure 11. Ground state energy of (55) as a function of g . Upper region.

different spectral problems exist. However, it might appear surprising in the framework of Mostafazadeh’s approach where the crypto-Hermitian Hamiltonian is obtained by a non-unitary rotation out of Hermitian \tilde{H} representing a quite definite series in g and hence the spectrum of H and of \tilde{H} represents a quite definite series in g . For example, the ground state energy of the system

$$H = \frac{p^2 + x^2}{2} - igx^5 \tag{55}$$

is

$$E_0 = \frac{1}{2} + \frac{449g^2}{32} + O(g^4). \tag{56}$$

The resolution of this paradox is the following. Seemingly, only *one* of the spectral problems (49) associated with the Hamiltonian (55), the problem defined in the sector including the real axis, can be safely treated in the framework of Mostafazadeh’s approach. The ground state energy is plotted in figure 11 as a function of g . Indeed, the spectrum tends to the oscillator spectrum when $g \rightarrow 0$. It is not seen on the plot, but for very small g , starting from $g \approx 0.01$ – 0.02 , the numerical values of the energies agree with the perturbative evaluation (56).

The solution for another spectral problem at the vicinity of the rays $\alpha = -3\pi/14$ and $\alpha = 17\pi/14$ behaves in a different and rather unexpectedly different way. For very small g , the spectrum is transformed, indeed, to the oscillator spectrum, but this transformation occurs in a very nontrivial manner. When g goes down, the energies of all the states go down in such a way that the energy of the ground state gets closer and closer to the energy of the first excited state. At some critical value of the coupling $g_* \approx 0.03717$, their energies coincide,

$$E_0(g_*) = E_1(g_*) \approx 0.484.$$

At still lower values of g , the energies should become complex. On the other hand, the second excited state goes down and down with decreasing of g and approaches the ground state oscillator energy without adventures, $E_2(g \rightarrow 0) \rightarrow 1/2$ (see figure 12).

To be more precise, there are no adventures in a sense that there is no phase transition and the state exists at any g and has real energy. But the asymptotics is reached only at rather small couplings. The energy of the second excited state finds itself at the vicinity of $E = 1/2$ only at $g \approx 0.01$. Now, $E_2(0.01) \approx 0.46$ and does not coincide with the perturbative expansion (56). It is not excluded that at still smaller values of coupling, $g \approx 0.001$, the perturbative asymptotics (56) finally shows up. To see whether it is true or not, a more careful numerical study is required.

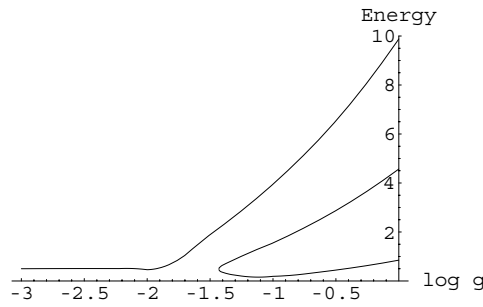


Figure 12. Three first levels of (55) as a function of g . Lower region.

The third and fourth excitations of the Hamiltonian (55) coalesce and their energies cease to be real at $g_{**} \approx 0.007$ (the energy is $E_{**} \approx 1.37$ at this point), while the fifth excitation approaches the first oscillator excitation $E = 3/2$ at very small values of g . One can suggest that this pattern holds also further up: the sixth and seventh excitations of the mixed Hamiltonian coalesce and their energy becomes complex at some very small g_{***} , while the eighth excitation approaches the second oscillator excitation $E = 5/2$, and so on. We thus observe an infinite sequence of ‘phase transitions’ in the coupling¹⁷.

This analysis shows that the Hamiltonian (55) is crypto-Hermitian for all couplings in the upper sectors in figure 10, but, in the lower sectors, it is true only for not too small $g > g_*$. When $g < g_*$, a pair of complex conjugate eigenvalues should appear. For $g < g_{**}$, there are two such pairs, etc. It would be very interesting to see these complex eigenvalues explicitly. Unfortunately, it is not so easy to do it with our methods—the spectral problems of the type (44) and (49) make sense only for real energies—the boundary condition

$$\operatorname{Re} \left(\frac{\partial \Psi / \partial s}{\Phi \Psi} \right) \Big|_{s=0} = 0$$

was derived under the assumption that $\Psi(z)$ and $\Psi^*(-z)$ satisfy the same Schrödinger equation, which is only true when E is real. A special study of this issue is required.

5. Discussion and outlook

Crypto-Hermitian systems have many common features with the systems involving higher derivatives. In both cases, Hermiticity of the Hamiltonian and unitarity of the evolution operator seem to be lost, but, if treating the problem properly, they are often restored. There exists also a more concrete relationship between two kind of systems. We have seen that the real part H of the complexified Hamiltonian (see equations (9) and (24)) is never positive definite and may give rise to ghosts. The same is true for higher-derivative theories. Actually, the canonical Hamiltonians of the latter have a rather similar form with not positive definite

¹⁷ This kind of transition when a pair of real eigenstates of a boundary problem coalesce and become complex is a known phenomenon [7]. Its essence is clearly seen in a trivial example. The matrix

$$A = \begin{pmatrix} 1 & 1 \\ \alpha & 1 \end{pmatrix}$$

has a pair of close real eigenvalues at small positive α and a pair of complex conjugated eigenvalues for $\alpha < 0$. When $\alpha = 0$ (the exceptional point), the matrix represents a nondiagonalizable Jordan block. An infinite set of such transitions in the parameter ϵ for the problem $V(x) = x^2(ix)^\epsilon$ was observed in [9]. We observed a similar phenomenon in a completely different physical context: it happens that some *domain wall* solutions in supersymmetric gauge theories disappear when mass of the matter fields exceeds certain critical values [26].

kinetic term [23]. The resemblance between the supersymmetric system analyzed in [23] and the problem considered here is even more striking. A system of the type (24) involves besides H the integral of motion G , and we are interested with the sector $G = 0$. The system studied in [23] (the bosonic part of its Hamiltonian is

$$H = pP - DV'(x), \quad (57)$$

where (p, x) and (P, D) are two pairs of canonic variables and superpotential $V(x)$ is an arbitrary function) also possesses an extra integral of motion $N = P^2/2 - V(x)$. In the sector with a particular value of N (including $N = 0$), the spectrum is discrete involving positive and negative energies.

The latter is also true for the spectrum (21) of complexified oscillator when the constraint $G = 0$ is imposed on the quantum states as in equation (18). The Dirac quantum problem (18) is more naturally posed than other quantum problems associated with the classical system in hand. This problem is easily solved in the oscillator case, but, for the potential iz^3 , this is a *difficult* numerical problem, and we leave it for further studies. One can speculate that its spectrum involves positive and negative energies, as the spectrum of the complexified oscillator and the spectrum of the Hamiltonian (57) do. However, it is an open question at present whether the problem (18) *makes sense* for potentials more complicated than z^2 . As we have seen in [23], the Hermiticity of the Hamiltonian (57) and the unitarity of the corresponding evolution operator are corollaries of the fact that classical trajectories of this system are benign enough: there are no collapsing or runaway trajectories where a singularity is reached at finite time. On the other hand, for the systems (4), runaway classical trajectories exist. For sure, not all the trajectories associated with the systems (4) are runaway trajectories. There are also closed orbits, and a hope that the problem (18) is well defined is associated with their existence. The presence of runaway trajectories may spoil the brew, however.

Runaway trajectories *definitely* spoil the brew for the quantum problems obtained by resolving the gauge constraint $G = 0$ at the classical level. This procedure gives benign sensible Hamiltonians for the complexified oscillator. However, the Hamiltonian (36) thus obtained is not Hermitian and unitarity is lost too.

There are, however, Hermitian and unitary quantum problems associated with the Hamiltonians (4) and (7). For one of such problems corresponding to the potential $x^2/2 - igx^5$ in the sectors below the real axis, we discovered a rather interesting and nontrivial phenomenon: when the coupling constant g is decreased, certain quantum states coalesce and disappear from the physical (real energy) spectrum. The number of such phase transitions is infinite, which reminds an infinite number of phase transitions in ϵ for the potential $x^2(ix)^\epsilon$ observed in [9]. Another phenomenon that comes to mind in this respect is the marginal stability curves in $N = 2$ SYM theory and other supersymmetric systems [27]. When crossing these curves, quantum states may appear and disappear. However, the mechanism for this is quite different there.

Let us make a somewhat unusual conclusion listing again not the results obtained (that was done above), but rather the points which are not yet clear.

- (1) It is not clear whether the spectral problem (18) is well posed for the potential (4) and, if yes, what is its spectrum. Is the evolution operator unitary?
- (2) It is not clear whether one can formulate the spectral problems with discrete spectrum in the dashed region in figures 9 and 10 by resolving the gauge constraint at the classical level with a clever gauge choice.
- (3) It is not clear why, in contrast to the complex oscillator case, we have not found for the potential (4) a spectral problem involving only negative energy states (the cryptoghosts!)

and related to the sets of classical orbits with negative energies. Can such problem be formulated?

The final remark is that crypto-Hermitian systems may prove to be something more than a formal mathematical exercise. They can bear relevance for physics. Our own interest in these problems stems mainly from their relationship to higher-derivative systems. And we believe (the arguments were presented in [28]) that the undiscovered yet fundamental Theory of Everything is a higher-derivative field theory (*not* string theory) living in higher-dimensional spacetime.

Acknowledgments

I am indebted to P Dorey, L Mezincescu and D Robert for useful discussions.

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